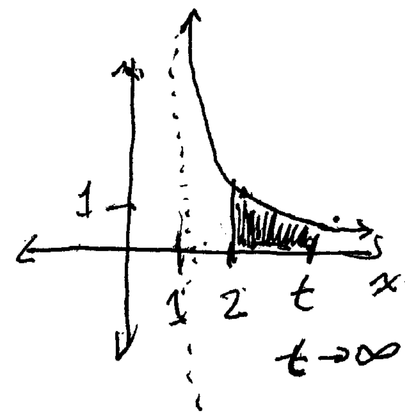
PROJECTED WRITTEN NOTES FROM THE M408 Lecture on Tuesday, January 30, 2024, on Sec 7.8: Improper Integrals, on Sec 11.1: Sequences, and Sec 11.2: FIRST NOTES ON SERIES

CLASS #5

Improper Integrals

Last Time

$$\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\,dx$$



It was shown that this limit exists and it equals 1

So "The Improper Integral is convergent and has the value 1.

It is an integral over $[2, \infty)$

IMPROPER INTEGRALS

Type I: [OVER AN UNBOUNDED INTERVAL]

$$\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^{t} f(x)\,dx$$ when the limit exists!

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_t^{b} f(x)\,dx$$ when the limit exists!

When the limit exists for both and with $a=b$; Then $\rightarrow$ $$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx.$$

Type II: [When function $f(x)$ fails to be continuous
A: at the left endpoint a
B: at the right endpoint b
C: at some number c with $a<c<b$]

A: $$\int_a^{b} f(x)\,dx = \lim_{t \to a^+} \int_t^{b} f(x)\,dx$$ when limit exists.

B: $$\int_a^{b} f(x)\,dx = \lim_{t \to b^-} \int_a^{t} f(x)\,dx$$ when limit exists.

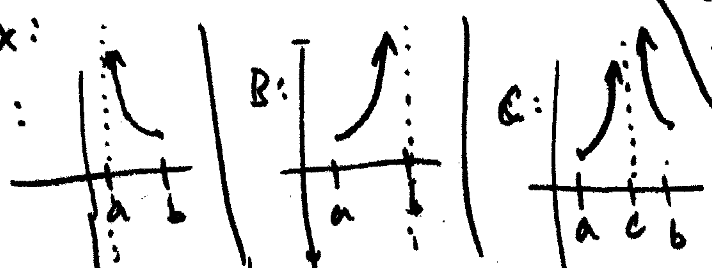
C: $$\int_a^{b} f(x)\,dx = \int_a^{c} f(x)\,dx + \int_c^{b} f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^{t} f(x)\,dx + \lim_{t \to c^+} \int_t^{b} f(x)\,dx$$

When these limits exist.

Ex:

A:



Type II Improper Integrals

Over a bounded interval $[a,b]$ such that

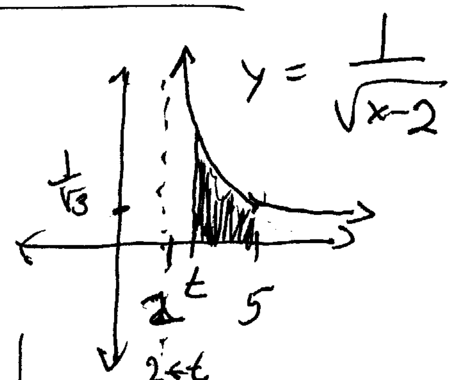
(1) $f(x)$ is not continuous at $x=a$,

(2) $f(x)$ is not continuous at $x=b$,

(3) $f(x)$ is not continuous at some number $x=c$ where $a<c<b$.

A Type II Example

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$



let $u = x-2$
$du = dx$

when $x=t$,
$u = t-2$
when $x=5$, $u=3$

$$= \lim_{t\to 2^+} \int_{t-2}^{3} \frac{1}{\sqrt{u}}\,du$$

y is not defined at $x=2$.

$$= \lim_{t\to 2^+} \left(2\sqrt{u}\right)\Big|_{t-2}^{3}$$

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{u}+C$$

$$= \lim_{t\to 2^+} \left(2\sqrt{3} - 2\underbrace{\sqrt{t-2}}_{\to 0}\right) = 2\sqrt{3}$$

The Limit exists

The Improper Integral $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ is Convergent and has the value $2\sqrt{3}$.

Sequences (Sec 11.1)

A sequence $\{a_n\}_{n=1}^{\infty}$ is an infinitely long list of numbers.

$a_1, a_2, a_3, a_4, \ldots$

↖ The Third Term of the Sequence.

Ex: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is the list: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, ...

$n=1$ $\quad n=2 \quad n=3 \ldots$

$\boxed{\text{Here, } a_n = \frac{n}{n+1}}$

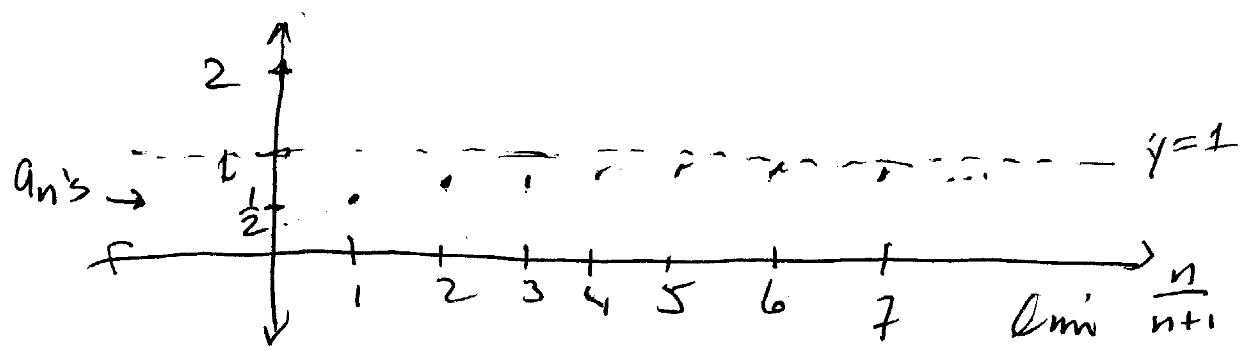
Definition:

The Sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ has a limit L if,

as $n \to \infty$, $\left|L - \frac{n}{n+1}\right| \to 0$, and, if so,

we say "$L = \lim_{n\to\infty} \frac{n}{n+1}$". Similarly, for any sequence $\{a_n\}_{n=1}^{\infty}$

Here, $\lim_{n\to\infty} \frac{n}{n+1} = L = 1$.



$\lim_{n\to\infty} \frac{n}{n+1} = 1$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{\cancel{n}(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = \frac{1}{1} = 1$$

You might be tempted to use L'Hospital's Rule here, But you can't apply L'Hospital's Rule to a Sequence directly. What you must do is to define a function of x, say "let $f(x) = \frac{x}{x+1}$".

$$\lim_{n\to\infty} \frac{n}{n+1} = \underbrace{\lim_{x\to\infty} \frac{x}{x+1}}_{\text{Form: } \frac{\infty}{\infty}} \overset{L.R.}{=} \lim_{x\to\infty} \frac{1}{1} = 1$$

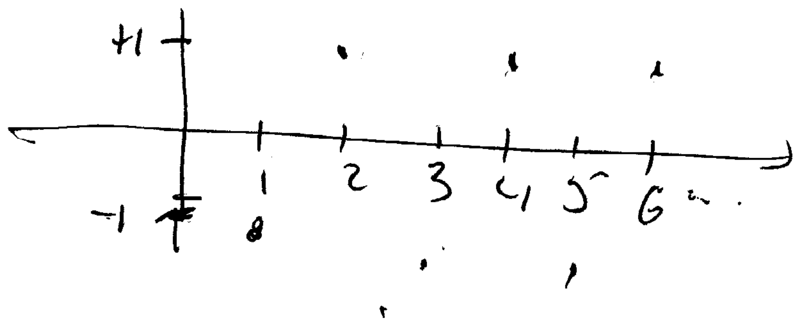
Another EXAMPLE:

$$\lim_{n\to\infty} \frac{\sqrt{n+3}}{\sqrt{4n+5}} = \lim_{n\to\infty} \frac{\sqrt{n(1+3/n)}}{\sqrt{n(4+5/n)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\sqrt{1+\overset{\to 0}{3/n}}}{\cancel{\sqrt{n}}\sqrt{4+\underset{\to 0}{5/n}}} = \frac{\sqrt{1}}{\sqrt{4}} = \underline{\underline{\frac{1}{2}}}$$

For some sequences, the limit Does Not Exist.

Ex: $\{(-1)^n\}_{n=1}^{\infty}$ = " $-1, 1, -1, 1, -1, 1, -1, 1 \ldots$



$\lim\limits_{n\to\infty} (-1)^n$ D.N.E.

The Sequence is Divergent

A sequence FACT

For a fixed number r,

The sequence $\{r^n\}_{n=}^{\infty}$ is $\begin{cases} \text{Convergent if: } -1 < r \le 1 \\ \text{Divergent if } r \le -1 \\ \quad \text{or } r > 1. \end{cases}$

and $\lim\limits_{n\to\infty} r^n = 0$ when $-1 < r < 1$

and $\lim\limits_{n\to\infty} r^n = 1$ when $r = 1$.

Ex: $\lim\limits_{n\to\infty} (0.72)^n = 0$ since $-1 < 0.72 < 1$.

The Series $\sum_{n=1}^{\infty} a_n$.

Given a sequence $\{a_n\}_{n=1}^{\infty}$ (The Sequence of Terms),
we form another sequence $\{s_n\}_{n=1}^{\infty}$
("Called the Sequence of Partial Sums" (SOPS))
as follows:

$$\{s_n\}_{n=1}^{\infty} = \text{"}s_1, s_2, s_3, s_4, \ldots\text{"} \quad \text{where } s_n = \sum_{k=1}^{n} a_k.$$

$$\{a_n\}_{n=1}^{\infty} = \text{"}a_1, a_2, a_3, a_4, a_5, \ldots\text{"}$$

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$\vdots \quad \text{ETC...}$$

$$\sum_{n=1}^{\infty} a_n = \{s_n\}_{n=1}^{\infty}$$

The result S.O.P.S. is
what the series $\sum_{n=1}^{\infty} a_n$ really is.

What is $\sum_{n=1}^{\infty} (0.1)^n = \sum_{n=1}^{\infty} a_n$ where $a_n = (0.1)^n$?

The Sequence of terms here is $\{a_n\}_{n=1}^{\infty} = \{(0.1)^n\}_{n=1}^{\infty}$.

Sequence of Partial Sums $\left.\right\} =$

$$S_1 = 0.1$$

$$S_2 = (0.1) + (0.1)^2 \approx 0.11$$

$$S_3 = (0.1) + (0.1)^2 + (0.1)^3 = 0.111$$

$$S_4 = \cdots = 0.1111$$

$\vdots$ etc.

$$\sum_{n=1}^{\infty} (0.1)^n = \text{"} \underset{S_1}{0.1}, \underset{S_2}{0.11}, \underset{S_3}{0.111}, \underset{S_4}{0.1111}, \ldots \text{"}$$

It turns out that $\lim_{n\to\infty} S_n = \frac{1}{9} = \sum_{n=1}^{\infty} (0.1)^n$.

When $\lim_{n\to\infty} S_n$ exists, we say "The Series $\sum_{n=1}^{\infty} a_n$ is <u>Convergent</u> (C).

When $\lim_{n\to\infty} S_n$ DNE, we say "The Series $\sum_{n=1}^{\infty} a_n$ is Divergent (D).

If the series $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\lim_{n\to\infty} S_n = S,$$

then we say "S is the summation (or sum) of the series" and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

The Test for Divergence

Given a series $\sum_{n=1}^{\infty} a_n$, if you consider the sequence of Terms $\{a_n\}_{n=1}^{\infty}$ and you find that

$\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ D.N.E., then

The Series $\{S_n\}_{n=1}^{\infty}$, a.k.a. $\sum_{n=1}^{\infty} a_n$, is Divergent.

Ex: Is the Series $\sum_{n=1}^{\infty} \frac{n}{4n+1}$ C or D?

Sol'n: Consider the Sequence of terms $\left\{\frac{n}{4n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{4n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{\cancel{n}(4+\frac{1}{n})} =$$

$$= \lim_{n\to\infty} \frac{1}{4+\underbrace{\tfrac{1}{n}}_{\to 0}} = \frac{1}{4} \neq 0.$$

The Series $\sum_{n=1}^{\infty} \frac{n}{4n+1}$ is Divergent by the Test for Divergence because $\lim_{n\to\infty} \frac{n}{4n+1} = \frac{1}{4} \neq 0$.